A faithful tensor space representation for the blob algebra

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1 Introduction

The blob algebra is a diagram algebra extending the Temperley–Lieb algebra in a fairly natural way, which has a number of very nice properties (see [3] for a review). Some time ago Martin and Woodcock [13] stumbled across a curious 'tensor space' representation of the blob algebra, which turns out [12] to be a full tilting module [5] in quasihereditary specialisations [2, 4]. This raises the possibility of some intriguing new developments in invariant theory (see [12] for a discussion). In the study of affine Hecke algebra representation theory it is also useful for technical reasons (see [3, 8]), to study the blob algebra, and the tensor space representation, in arbitrary specialisations, including *non*-quasihereditary cases. In particular it is useful to know if the tensor space representation is *faithful* in arbitrary specialisations. In this paper we answer this question in the affirmative.

We begin by assembling the machinery we will need in the more familiar context of the Temperley–Lieb (TL) algebra. The ordinary tensor space representation here [14, 1, 9] was shown to be faithful a long time ago [11, 6], and we use a similar method to [11] here. However we are able to implement it in such a way that it is applicable to representations subject only to a relatively flexible set of conditions. Using this flexibility, we are eventually able to apply the method to the blob algebra, thus obtaining a sufficient condition for blob representations to be faithful.

In the final section we recall the construction of the blob tensor space representation, from which it is evident that it satisfies this condition.

In this paper K is a ring, x an invertible element in K, $q = x^2$, and $[n] = q^{n-1} + q^{n-3} + \ldots + q^{1-n}$. Define T_n^K to be the K–algebra with generators $\{1, U_1, \ldots, U_{n-1}\}$ and relations

$$U_i U_i = (q + q^{-1}) U_i \qquad (1)$$

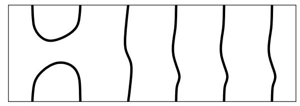
$$U_i U_{i\pm 1} U_i = U_i \qquad (2)$$

$$U_i U_j = U_j U_i \qquad (|i-j| \neq 1) \qquad (3)$$

2 Temperley–Lieb shenanigans

For $n + m$ even, an (n, m) TL diagram is a rectangular frame with n nodes on the northern and m nodes on the southern edge; the $n + m$ nodes are connected in

pairs by non–touching lines in the plane interior to the frame. Two such diagrams are identified if they partition the set of nodes into pairs in the same way. The set of such diagram is denoted $\mathcal{D}(n,m)$. Label the northern nodes $1, 2, .., n$ and the southern nodes $1', 2', .., m'$. Say $(ij) \in D$ if nodes i, j (primed, unprimed or mixed) are connected in diagram D. Write $1 \in \mathcal{D}(n,n)$ for the element such that $(ii') \in 1$ for all i. Write $\mathcal{U}_j \in \mathcal{D}(n,n)$ for the element such that $(j\, j{+}1), (j'\, (j{+}1)') \in \mathcal{U}_j$ and $(ii') \in \mathcal{U}_j$ for all $i \neq j, j+1$. For example

$\mathcal{U}_1 :=$ 

Define a product

$$\begin{array}{rcl} \mathcal{D}(n,m) \times \mathcal{D}(m,l) & \to & \mathcal{D}(n,l) \\ (D, D') & \mapsto & D{\circ}D' \end{array}$$

by first concatenating the diagrams D, D' in such a way that the i^{th} primed node of D meets the i^{th} unprimed node of D'. (Call this object $D|D'$.) These nodes are then discarded, leaving connections amongst the nodes of a resultant diagram in $\mathcal{D}(n,l)$. Note that $D|D'$ may have some closed loops, which we ignore in $D{\circ}D'$. However define a map

$$\begin{array}{rcl} \mathcal{D}(n,m) \times \mathcal{D}(m,l) & \to & \mathbb{N} \\ (D, D') & \mapsto & \zeta((D, D')) \end{array}$$

where $\zeta((D, D'))$ is the number of closed loops discarded above. Thus for example $\mathcal{U}_1{\circ}\mathcal{U}_1 = \mathcal{U}_1$ and $\zeta((\mathcal{U}_1, \mathcal{U}_1)) = 1$.

The *propagating number* $\#(D)$ of a diagram D is the number of lines of the form (ij') in D. Note that it is possible to cut a diagram from the western to the eastern edge in such a way that only these lines are cut, and they are cut once each. Let $\mathcal{D}^l(n,m)$ denote the subset of $\mathcal{D}(n,m)$ with propagating number l. Note that cutting as above defines a unique map

$$\begin{array}{rcl} \mathcal{D}^l(n,m) & \to & \mathcal{D}(n,l) \times \mathcal{D}(l,m) \\ D & \mapsto & (D^{\cup}, D_{\cap}) \end{array}$$

such that $D^{\cup}{\circ}D_{\cap} = D$.

Set $q = x^2$, $x \in K$, and define delta-function

$$\delta_{ab} = \begin{cases} 1 & a = b \\ 0 & \text{otherwise} \end{cases}$$

$\delta'_{ab} = 1 - \delta_{ab}$, and, for $a \in \mathbb{Z} \setminus \{0\}$

$$\mathrm{sign}(a) = \begin{cases} +1 & a > 0 \\ -1 & a < 0 \end{cases} .$$

Associate to each $D \in \mathcal{D}(n,m)$ a matrix $R_q(D)$ as follows. Rows are indexed by the set $\mathrm{seq}_n\{1,2\}$ of words in $\{1,2\}$ of length n. Columns are indexed similarly by

$\mathrm{seq}_m\{1,2\}$. For $v \in \mathrm{seq}_n\{1,2\}$ write v_i for the i^{th} term. Then

$$R_q(D)_{vw} = \left(\prod_{\substack{i<j,\\ (ij)\in D}} q^{\frac{\mathrm{sign}(v_i - v_j)}{2}} \delta'_{v_i v_j} \right) \left(\prod_{(ij')\in D} \delta_{v_i w_j} \right) \left(\prod_{\substack{i<j,\\ (i'j')\in D}} q^{\frac{\mathrm{sign}(w_i - w_j)}{2}} \delta'_{w_i w_j} \right) \tag{4}$$

For example, $R_q(1)$ is the unit matrix.

Proposition 1 *Suppose there are no closed loops in $D|D'$. Then for each pair u, v there exists a unique w giving rise to a non-vanishing summand in*

$$(R_q(D)R_q(D'))_{uv} = \sum_w R_q(D)_{uw} R_q(D')_{wv};$$

and

$$R_q(D)R_q(D') = R_q(D \circ D').$$

More generally,

$$R_q(D)R_q(D') = [2]^{\zeta((D,D'))} R_q(D \circ D'). \tag{5}$$

Proof: Fixing u, v and considering $\sum_w R_q(D)_{uw} R_q(D')_{wv}$ we have

$$\sum_w \left(\prod_{\substack{i<j,\\ (ij)\in D}} q^{\frac{\mathrm{sign}(u_i - u_j)}{2}} \delta'_{u_i u_j} \right) \left(\prod_{(ij')\in D} \delta_{u_i w_j} \right) \left(\prod_{\substack{i<j,\\ (i'j')\in D}} q^{\frac{\mathrm{sign}(w_i - w_j)}{2}} \delta'_{w_i w_j} \right)$$

$$\left(\prod_{\substack{i<j,\\ (ij)\in D'}} q^{\frac{\mathrm{sign}(w_i - w_j)}{2}} \delta'_{w_i w_j} \right) \left(\prod_{(ij')\in D'} \delta_{w_i v_j} \right) \left(\prod_{\substack{i<j,\\ (i'j')\in D'}} q^{\frac{\mathrm{sign}(v_i - v_j)}{2}} \delta'_{v_i v_j} \right)$$

Each delta-function factor corresponds to a line in $D|D'$ (the arguments correspond to the endpoints of the line). In particular each w_i appears in two delta-functions. Hence each delta-function (or complementary delta-function) involving w lies in a *chain* of one of a number of possible types. If there are no closed loops in $D|D'$ then those lines/deltas involving w must lie in chains which begin either in u or in v. For example we might have w_1, w_2 appearing in the form

$$\sum_{w_1, w_2} \delta_{u_1 w_1} q^{\frac{\mathrm{sign}(w_i - w_j)}{2}} \delta'_{w_1 w_2} \delta_{w_2 u_2} = q^{\frac{\mathrm{sign}(u_i - u_j)}{2}} \delta'_{u_1 u_2}$$

where the right hand side shows the result of performing the relevant summations. Since every w_i arises in this way, the complete sum may be replaced by precisely one term — up to powers of q, a product of delta functions involving u, v. Considering an individual chain involving w, if it is ultimately propagating then an equal number of (ij) lines from D' and $(i'j')$ lines from D are involved, contracting to a simple delta function. If it is ultimately within u then there must be one more (ij) line from D' than $(i'j')$ lines from D, and so on.

The general result follows by a similar argument. $\square$

Definition 1 *Two matrices M, N are* mask equivalent *if $M_{ij} = 0 \iff N_{ij} = 0$. Write $[M]$ for the equivalence class of M.*

Note,

$$[R_q(D)] = [R_{q'}(D)]. \tag{6}$$

We will write $R(D)$ for $R_q(D)$ (other choices of parameter will be written explicitly).

Proposition 2 *Provided there are no closed loops in $D|D'$, if $X \in [R(D)]$ and $Y \in [R(D')]$ then $XY \in [R(D \circ D')]$.*

Proof: The delta function structure of $R(D)_{vw}$ has now been overlain, in X_{vw}, with an arbitrary nonzero constant, $X_{vw} = k^X_{vw} R(D)_{vw}$, say. But since the delta function structure is the same, fixing u, v we still have only one value of w (w^{uv} say) producing a nonvanishing term in $\sum_w X_{uw}Y_{wv}$. Thus $\sum_w X_{uw}Y_{wv} = k^X_{uw^{uv}} k^Y_{w^{uv}v} \sum_w R(D)_{uw}R(D')_{wv}$. $\square$

3 Temperley–Lieb algebra

For K a ring and q a unit in K let $\mathcal{T}_n$ denote the Temperley–Lieb algebra, a K–algebra with basis $\mathcal{D}(n, n)$ and multiplication given by

$$D.D' = [2]^{\zeta((D,D'))} D \circ D'.$$

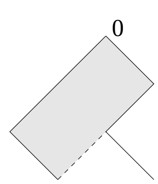
Thus from (5), R on $\mathcal{D}(n, n)$ extends to a representation of $\mathcal{T}_n$ (in fact the usual action on tensor space [14, 1]). The following two results are standard [7, 10].

Proposition 3 *$\mathcal{T}_n$ is generated by $\{1, \mathcal{U}_1, \mathcal{U}_2, .., \mathcal{U}_{n-1}\}$.*

Proposition 4 *$\mathcal{T}_n$ is isomorphic to the algebra with generators $\{1, U_1, U_2, .., U_{n-1}\}$ and relations as in equations 1 to 3, with isomorphism given by $\mathcal{U}_i \mapsto U_i$.*

The Pascal triangle may be viewed as a graph embedded in the plane. It has vertices arranged in layers called levels. Levels are indexed $0, 1, 2, ...$ Within level i vertices are indexed by 'column': $i, i-2, .., -i$. Thus a specific vertex may be labelled by (level,column)$= (i, i-2j)$. Edges are given by pairs of vertices: $((i, j), (i+1, j\pm 1))$. The 1–Pascal graph is the full subgraph on vertices with nonnegative column index.

Let $\mathcal{W}_i(n)$ be the set of walks of length n from (0,0) to (n, i) on the 1-Pascal graph. These walks may be represented in an obvious way by elements of $\mathrm{seq}_n\{1, 2\}$ (choose all such walks to start $1 \ldots$). Let $\mathcal{W}^2_i(n) = \mathcal{W}_i(n) \times \mathcal{W}_i(n)$ and $\mathcal{W}^2(n) = \cup_i \mathcal{W}_i(n) \times \mathcal{W}_i(n)$. Draw an element (a, b) of $\mathcal{W}^2(n)$ by drawing a and the image of b under reflection in the main vertical of the Pascal triangle. The *envelope* of $(a, b) \in \mathcal{W}^2_i(n)$ is the subset of the plane bounded by this drawing and the piecewise straight line from vertex (n, i) to $(n - i, 0)$ to $(n, -i)$. For example, the envelope of $(121, 112)$ is

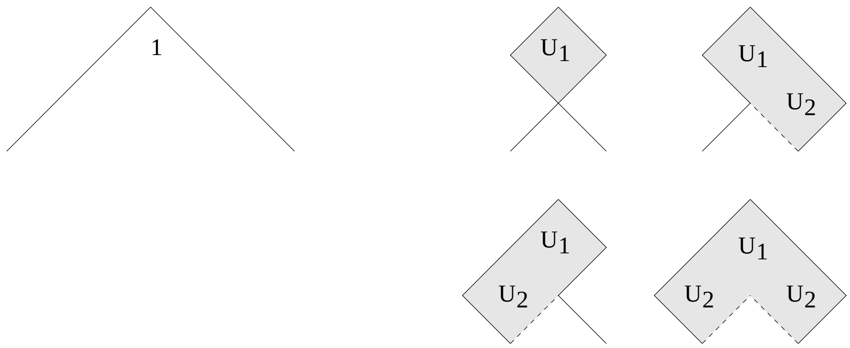


Partial order $\mathcal{W}^2_i(n)$ by $(a,b) \leq (c,d)$ if the drawing of (a,b) never leaves the envelope formed by (c,d). (We will also use the obvious underlying partial order on *single* walks. This partial order is a lattice, with lowest walk 1212..1211..1, and highest walk 11..122..2.) Extend to a partial order on $\mathcal{W}^2(n)$ by $(a,b) \leq (c,d)$ if the endpoint of a is (n,i), that of c is (n,j), and $i < j$.

The envelope of (a,b) may be tiled in an obvious way with diamonds (squares oriented at 45^o) of side length 1. Form a map

$$w : \mathcal{W}^2(n) \to T_n$$

by scanning the tiling of (a,b) from top to bottom, left to right, and writing U_i for each tile with base at row position i. Example:



Proposition 5 *None of the elements in $w(\mathcal{W}^2(n))$ has a diagram representation with a closed loop. As diagrams $w(\mathcal{W}^2(n)) = \mathcal{D}(n,n)$.*

Proof: see for example [10, §6.5].

If a is a walk or sequence with subsequence 21, with the 1 in the i^{th} position, then write a^i for the same sequence except that the subsequence is replaces by 12. Note that $(a^i, b) > (a,b)$ for any b, and that $w((a^i,b)) = U_i w((a,b))$.

Proposition 6 *(1) If $a = u$ and $b = v$ (confusing walks and sequences as above) then*

$$R(w((a,b)))_{uv} \neq 0.$$

(2) If

$$R(w((a,b)))_{uv} \neq 0$$

then $(a,b) \geq (u,v)$.

Proof: First note that (1) is true for the lowest walk pair in each lattice $\mathcal{W}^2_i(n)$ by an explicit calculation. For example, in bra–ket notation

$$\langle 1212..|U_1U_3|1212..\rangle = \langle 1212..|U_3|q1212.. + 2112..\rangle =$$

$$\langle 1212..||q^2 1212.. + q1221.. + q2112.. + 2121..\rangle = q^2.$$

NB, (2) is the same as: if $(a,b) \not\geq (u,v)$ then $R(w((a,b)))_{uv} = 0$. Thus we may approach the whole proposition by working through various cases of (a,b) and (u,v). For our first case, suppose that a ends at $(n, n-2i)$ and u at $(n, n-2j)$ with $i > j$. In this case $(a,b) \not\geq (u,v)$ by virtue of their being in *different* lattices. Consider the lowest walk pair ((a^o, a^o), say) in the lattice containing (a,b). This has $w((a^o, a^o)) =$

$U_1U_3..U_{2i-1}$. Given that $U_i|..11..\rangle = 0$, a simple sorting argument shows that there must be at least i 2s in the sequence u for there to be a nonzero matrix element. In our case, however, there are precisely j 2s. Thus $w((a^o, a^o)) = U_1U_3..U_{2i-1}$ vanishes on the whole permutation block associated to us of this type. But the rest of $w(\mathcal{W}^2_{n-2i}(n))$ is in the ideal generated by $w((a^o, a^o)) = U_1U_3..U_{2i-1}$, so the image of every pair in $\mathcal{W}^2_{n-2i}(n)$ vanishes.

It remains to deal with cases in which both (a, b) and (u, v) are drawn from the same lattice $\mathcal{W}^2_k(n)$ (some k). We work by induction on the lattice $w(\mathcal{W}^2_k(n))$. That is, we suppose the proposition holds as regards all pairs below (a, b), and all pairs (u, v). Then in particular it holds for some pair (a, c) such that $c^i = b$. We have

$$\langle a|w((a, b))|b\rangle = \langle a|w((a, c^i))|c^i\rangle = \langle a|w((a, c))U_i|c^i\rangle$$

$$= \langle a|w((a, c))|qc^i + c\rangle = q\langle a|w((a, c))|c^i\rangle + \langle a|w((a, c))|c\rangle$$

Since $(a, c) \not\geq (a, c^i)$ the first term vanishes by the inductive hypothesis; the second does not, also by the inductive hypothesis. Thus (1) holds provided the inductive step for (2) holds.

As regards (2), first note that the base case is again straightforward: the lowest pair in the lattice gives $U_1U_3..$, which kills every sequence except the corresponding lowest one (the first step is always 1, U_1 kills the sequence unless the second step is 2; the third step is now forced to be 1, and U_3 kills the sequence unless the fourth is 2; and so on). To prove the induction consider $\langle u|w((a, b))$. By the left–right symmetry of our problem we are done if we can show this vanishes when $u \not\leq a$, so we restrict to such cases. We may assume WLOG that there is some d and some i such that $a = d^i$, whereupon

$$\langle u|w((a, b)) = \langle u|w((d^i, b)) = \langle u|U_i w((d, b)).$$

(NB, $(d, b) < (a, b)$ so the inductive assumption holds for (d, b) with all (u, v).) If $u_i = u_{i+1}$ the last expression vanishes and we are done. Otherwise, we have

$$\langle u|U_i w((d, b)) = \left(q^{\pm 1}\langle u| + \langle u^{(i)}|\right) w((d, b))$$

where $u^{(i)}$ may be either higher or lower than u, depending on whether $u = ..12..$ or $..21..$ in the i^{th} position. Since $u \not\leq a$ and $a > d$ we have $u \not\leq d$ and the first term vanishes by the inductive assumption (note that the ket part is not needed for this). If $u^{(i)} > u$ then the second term vanishes similarly. If $u^{(i)} < u$ then the i^{th} and $i + 1^{th}$ elements of both $u^{(i)}$ and d are 21. Thus $u \not\leq a$ implies $u^{(i)} \not\leq d$. $\Box$

Proposition 7 *The matrices $R(w(\mathcal{W}^2(n)))$ are a linearly independent set. The representation R is faithful.*

Proof: Pick a total order consistent with the partial order. As we run up through the order there is, for each element, a matrix element which becomes nonzero first for that element.

It is a straightforward exercise to show that $|\mathcal{W}^2(n)| = \mathrm{Rank}(T_n)$. $\Box$

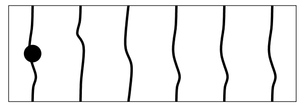
Since the proof above uses only the occurences of nonzero matrix elements we have, by the same token, a result on mask equivalent matrices:

Proposition 8 *Any set $\{X_D \in [R(D)] \mid D \in w(\mathcal{W}^2(n))\}$ is linearly independent.*

4 The blob algebra

In what follows it is convenient to shift the indices on the generators of T_{2n} so that they run $U_{-n+1}, U_{-n+2}, .., U_0, .., U_{n-1}$. As before, the matrices $R(U_i)$ provide a representation of T_{2n}.

A line in a TL diagram is *exposed* if it may be deformed to touch the western edge of the frame. A *blob diagram* is like a TL diagram, except that any exposed line may be decorated with a blob. For example

$e :=$ 

Write $\mathcal{D}^b(n,m)$ for the extension of $\mathcal{D}(n,m)$ to include decorated diagrams in this way. Blob diagram composition is like TL diagram composition, except that:
∘ two blobs on the same line may be replaced by one blob and a factor $\delta_e \in K$, and
∘ a closed loop with a blob is replaced by a factor $\gamma \in K$ instead of [2].
Thus the blob algebra b_n with basis $\mathcal{D}^b(n,n)$ and this composition has three parameters, q, γ and δ_e. (Over a field, $\delta_e \neq 0$ may be rescaled to 1 without loss of generality, but this point need not concern us here.)

The proof of the following is straightforward.

Proposition 9 *b_n is generated by $\{1, e, \mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{n-1}\}$. In particular, every element of $\mathcal{D}^b(n,n)$ may be expressed as a word in this set.*

A word in the generators of b_n (resp. T_n) is *loop free* if its blob (resp. TL) diagram has no loops. This includes reduced words and words whose diagrams differ from those of reduced words only by ambient isotopies. Let B_n be a set of loop free words in the generators of b_n which, as blob diagrams, form the diagram basis of b_n. Let f be the map from words in the b_n generators to words in the T_{2n} generators given piecewise by $f(e) = U_0$ and $f(\mathcal{U}_i) = U_{-i}U_i$. Note that this takes loop free words to loop free words, but is not an algebra map. It induces an injective set map from the diagram basis of b_n into $\mathcal{D}(2n, 2n)$.

Definition 2 *A representation ρ of b_n with $\rho(e) \in [R_r(U_0)]$, and $\rho(\mathcal{U}_i)$ of the form X_iY_i, where $X_i \in [R_s(U_{-i})]$ and $Y_i \in [R_t(U_i)]$ for some r, s, t, is called a* mirror *representation.*

(NB, by equation 6 the choice of r, s, t is actually irrelevant to this statement; the point of introducing mask equivalence is that it does not differentiate r, s, t, but preserves 'enough' of the structure of diagram composition, as in proposition 2, to allow us to prove the following.)

Proposition 10 *Any mirror representation of b_n is faithful.*

Proof: Consider the set of matrices $\{\rho(w) \mid w \in B_n\}$. It follows from proposition 2 that $\rho(w)$ is mask equivalent to $R(f(w))$.[1] For example, $\rho(U_i) = X_iY_i \in [R(U_iU_{-i})]$. At the level of diagrams, the map f from B_n to the set of TL diagrams is injective. That is, the set $f(B_n)$ of T_{2n} words is, as a set of diagrams, a subset of $\mathcal{D}(2n, 2n)$. It therefore follows from proposition 8 that the set of representation matrices for the diagram basis of b_n is linearly independent. □

[1]The words $f(w)$ correspond to TL diagrams which are left–right symmetric. Indeed all symmetric diagrams in $\mathcal{D}(2n, 2n)$ may be obtained in this way (there are $\frac{(2n)!}{n!n!} = |f(B_n)|$ of them).

5 Mirror representations

We recall the representation ρ_0 of b_n defined in [13, §6.1]: As explained in [3], the most interesting unanswered questions about b_n concern certain cases in which (γ, δ_e) can be written in the form $([m-1]\alpha, [m]\alpha)$ for some $m \in \mathbb{Z}$ and some scalar α. (For example, b_n is not quasihereditary in general over a field in which $[2]=0$ and $\gamma=0$ and $\delta_e=1$.) Accordingly we recall the representation ρ_0 in an integral form suitable for passing to such cases.

Set

$$\mathcal{U}^q(\chi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & \chi \end{pmatrix}$$

and $\mathcal{U}^q = \mathcal{U}^q(0)$.

Let $V_2 = K^2$. Fix n and let $M_2^r(U_i) \in \mathrm{End}(V_2^{\otimes 2n})$ be a matrix acting trivially on every tensor factor except the i^{th} and $(i+1)^{th}$, where it acts as $-\mathcal{U}^r$. (Thus $M_2^r(U_i) = R_r(U_i)$ for $U_i \in T_{2n}$ with $q=r$.)

Suppose there is an element $a \in K$ such that $a^4 = -1$. Then $a^2 + a^{-2} = 0$. Fix m such that $q^m \in K$ and set

$$r = a^2 q^m$$

$$s = a^5 x$$

$$t = a^3 x$$

Let $b_n^{\mathbb{Z}[q,q^{-1}]}(q,m)$ be $\mathbb{Z}[q,q^{-1}]\mathcal{D}^b(n,n) \subset b_n$ with $\gamma = q^{m-1} - q^{-m+1}$ and $\delta_e = q^m - q^{-m}$, a $\mathbb{Z}[q,q^{-1}]$–algebra. Then there is an algebra homomorphism

$$\rho_0 : b_n^{\mathbb{Z}[q,q^{-1}]}(q,m) \longrightarrow End_{\mathbb{Z}[a,x,x^{-1}]}(V_2^{\otimes 2n})$$

given by

$$\rho_0 : e \quad \mapsto \quad a^{-2} M_2^r(U_n) \tag{7}$$

$$\rho_0 : U_i \quad \mapsto \quad M_2^s(U_{n-i}) M_2^t(U_{n+i}). \tag{8}$$

Comparing with definition 2 and (4) we see that ρ_0 is a mirror representation, and hence faithful.

6 Discussion

In [11] corresponding statements to proposition 7 are proved for each of the ordinary Hecke algebra quotients $\mathrm{End}_{U_q sl_N}(V_N^{\otimes n})$, $V_N = K^N$ (explicitly for $K = \mathbb{C}$, since this is a Physics reference, but the restriction is not forced). It would be extremely desirable to generalise the blob version in an analogous way, since the generalised blob algebras provide direct information about affine Hecke representation theory [13]. So far not even a candidate representation is known!

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